Exponentiable Spaces

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Abstract

Notes for a talk I gave. I apologise for the errors.

Definition 1 *A space X is said to be* ***exponentiable*** *if the functor*

$$Top \xrightarrow{(-)\times X} Top, \qquad Z \mapsto Z \times X \tag{0.1}$$

has a right adjoint R_X. □

We write $C(X,Y)$ for the set of continuous maps $X \to Y$. If X is exponentiable, then for any spaces Y, Z there are natural bijections $C(Z \times X, Y) \cong C(Z, R_X(Y))$. Taking $Z = *$ we obtain $C(X,Y) = R_X(Y)$ (as sets). Thus $R_X(Y)$ is obtained by placing a suitable topology on $C(X,Y)$. When given a topology τ on this set we will write $C_\tau(X,Y)$ for the resulting space.

Definition 2 *Let X, Y be spaces and τ a toplogy on $C(X,Y)$. The topology τ is said to be;*

(1) ***splitting***[1] *if for every space Z and every continuous $f : Z \times X \to Y$, the adjoint $f^\# : Z \to C_\tau(X,Y)$ is continuous.*

(2) ***cosplitting***[2] *if for every space Z and every continuous $f : Z \to C_\tau(X,Y)$, the adjoint $f^\flat : Z \times X \to Y$ is continuous.*

(3) ***exponential***[3] *if it is both splitting and cosplitting.* □

Example 0.1 (1) The indiscrete topology on $C(X,Y)$ is always splitting, while the discrete topology is always cosplitting. In particular both splitting and cosplitting topologies always exist.

[1]Engleking calls this **proper**. Escardó-Heckman call this **weak**.

[2]Engelking calls this **admissible**. Goubault-Larrecq and Dugunji call this **conjoining**. Kelly and Nagata call this **jointly continuous**. Escardó-Heckman call this **strong**.

[3]Engelking calls this **acceptable**. Escardó-Heckman call this **even**.

(2) $C_{pw}(X,Y)$ denotes the topology of **pointwise convergence**. This topology is nothing but the subspace topology inherited from the inclusion $C(X,Y) \subseteq \prod_X Y$. To be consistent with later notation we write its subbasic opens as $\langle x, V\rangle = \{f \in C_{pw}(X,Y) \mid x \in f^{-1}(V)\}$, where $x \in X$ and $V \subseteq Y$ is open. The pointwise topology is splitting. It is almost never cosplitting: a map $Z \to C_{pw}(X,Y)$ is continuous if and only if its adjoint $Z \times X \to Y$ is continuous in each variable separately. Essentially the only cases to note are when Y is indiscrete or X is a point.

(3) We write $C_{ko}(X,Y)$ for the **compact-open topology** and let $\langle K, V\rangle = \{f \in C(X,Y) \mid K \subseteq f^{-1}(V)\}$, with $K \subseteq X$ compact and $V \subseteq Y$ open, denote its subbasic open sets. The compact-open topology is splitting. If X is locally compact, then $C_{ko}(X,Y)$ is cosplitting and hence exponential. We show below the converse: if X is a fixed *Hausdorff* space and $C_{ko}(X,Y)$ is exponential for each space Y, then X is locally compact. In fact, if X is Tychonoff and $C_{ko}(X,\mathbb{R})$ is exponential, then X is locally compact. Case in point, $\mathbb{Q}$ is not locally compact, and the compact-open topology on $C_{ko}(\mathbb{Q},\mathbb{R})$ is not cosplitting. □

Proposition 0.1 *For spaces X, Y, a topology τ on $C(X,Y)$ is cosplitting if and only if the evaluation map $ev : C_\tau(X,Y) \times X \to Y$ is continuous.*

Proof The evaluation is adjoint to the identity $C_\tau(X,Y) \to C_\tau(X,Y)$. If τ is cosplitting, then this is continuous. Conversely, if ev is continuous and a continuous $f : Z \to C_\tau(X,Y)$ is given, then $f^\#$ is the continuous function $f^\# : Z \times X \xrightarrow{f\times 1} C_\tau(X,Y) \times X \xrightarrow{ev} Y$. ∎

Proposition 0.2 *Let X, Y be spaces and σ, τ topologies on $C(X,Y)$.*

(1) If τ is splitting and $\sigma \subseteq \tau$, then σ is splitting. i.e. every topology weaker than a splitting toplogy is splitting.

(2) If τ is cosplitting and $\tau \subseteq \sigma$, then σ is cosplitting. i.e. every topology stronger than a cosplitting topology is cosplitting.

(3) If σ is splitting and τ is cosplitting, then $\sigma \subseteq \tau$. i.e. every splitting topology is weaker than every cosplitting topology.

It follows that there is always a largest splitting topology on $C(X,Y)$. Moreover there exists at most one exponential topology on $C(X,Y)$. When the exponential topology exists it coincides with the largest splitting topology, and is in this case also the smallest cosplitting topology.

Proof (1) and (2) are immediate and the last statements are a consequence of (3) and our previous observations. To prove (3) check that the chain of adjunctions

$$C(C_\sigma(X,Y), C_\sigma(X,Y)) \xleftarrow{\#} C(C_\sigma(X,Y) \times X, Y) \xrightarrow{\flat} C(C_\sigma(X,Y), C_\tau(X,Y)) \tag{0.2}$$

takes the identity on $C_\sigma(X,Y)$ to the comparison map $C_\sigma(X,Y) \to C_\tau(X,Y)$ which is induced by it. Since this map is continuous we get the statement. ∎

A first application: starting with a Hausdorff Y we have a Hausdorff product topology on $C_{pw}(X,Y)$. Since every cosplitting topology contains the pointwise topology, each cosplitting topology is Hausdorff. The same applies to any splitting topologies which contain it (eg. the compact-open topology).

Definition 3 *When the exponential topology on $C(X,Y)$ exists we will write Y^X for the corresponding space. Since the exponential topology is unique this notation is meaningful.* □

Proposition 0.3 *Let X, Y, Z be spaces. The following statements hold.*

(1) Let $f : Y \to Z$ be continuous. If $C_\sigma(X,Y)$ carries a cosplitting topology and $C_\tau(X,Z)$ carries a splitting topology, then the induced map $f_* : C_\sigma(X,Y) \to C_\tau(X,Z)$ is continuous.

(2) Let $g : X \to Y$ be continuous. If $C_\sigma(Y,Z)$ carries a cosplitting topology and $C_\tau(X,Z)$ carries a splitting topology, then the induced map $g^* : C_\sigma(Y,Z) \to C_\tau(X,Z)$ is continuous.

Proof (1) The map f_* is the adjoint of the continuous map $C_\sigma(X,Y) \times X \xrightarrow{ev} Y \xrightarrow{f} Z$. (2) The map g^* is adjoint to the composition $C_\sigma(Y,Z) \times X \xrightarrow{1\times g} C_\alpha(Y,Z) \times Y \xrightarrow{ev} Z$. ∎

Remark Similar methods establish the following.

(1) Assume that $C_\sigma(X,Y), C_\tau(Y,Z)$ carry cosplitting topologies and $C_\rho(X,Z)$ carries a splitting topology. Then the composition $C_\tau(Y,Z) \times C_\sigma(X,Y) \to C_\rho(X,Z)$ is continuous.

(2) If $C_\tau(Y,Z)$ and $C_\sigma(X, C_\tau(Y,Z))$ carry cosplitting topologies and $C_\rho(X \times Y, Z)$ carries a splitting topology, then the canonical map $C_\sigma(X, C_\tau(Y,Z)) \to C_\rho(X \times Y, Z)$ is continuous.

(3) If $C_\rho(X \times Y, Z)$ carries a cosplitting topology, and $C_\tau(Y,Z)$, $C_\sigma(X, C_\tau(Y,Z))$ carry splitting topologies, then the canonical map $C_\rho(X \times Y, Z) \to C_\sigma(X, C_\tau(Y,Z))$ is continuous.

(4) If $C_\tau(X,Y)$ is splitting, then the constants embedding $Y \to C_\tau(X,Y)$ is continuous. □

Corollary 0.4 *A space X is exponentiable if and only if the exponential object Y^X exists for each space Y.* ∎

The point is that while a necessary condition for X to be exponentiable is that exponential topologies exist on $C(X,Y)$ for all Y, it was not clear before how these topologies would assemble so as to make the assignment $Y \mapsto Y^X$ functorial. As it turns out, the exponential topology takes care of itself.

Let $\mathcal{E} \subseteq Top$ denote the full subcategory on the exponential spaces. It contains the empty space and the one-point space. Moreover it has finite products and all coproducts. On the other hand it does not have infinite products (consider the Baire space $\mathbb{N}^\omega \cong \mathbb{P}$), and it is not closed under passing to subspaces (consider $\mathbb{Q} \subseteq \mathbb{R}$) or quotients (consider $\mathbb{R}/\mathbb{N}$). The following is a slightly sharper statement of the last corollary.

Corollary 0.5 *The exponential defines a functor $\mathcal{E}^{op} \times Top \to Top$, $(X,Y) \mapsto Y^X$.* ∎

Example 0.2 (1) The **Isbell topology** $C_{Is}(X,Y)$ is splitting. Its definition follows. Denote by $\mathcal{O}_X$ the lattice of open subsets of X and call a subset $\mathbb{U} \subseteq \mathcal{O}_X$ **Scott open** if (i) whenever $U \in \mathbb{U}$ and $U \subseteq V \in \mathcal{O}_X$, we have $V \in \mathbb{U}$, and (ii) given any family $U_i \in \mathcal{O}_X$, $i \in \mathcal{I}$, with $\bigcup_{\mathcal{I}} U_i \in \mathbb{U}$, there are finitely many indices $i_1, \ldots, i_n$ such that $U_{i_1} \cup \cdots \cup U_{i_n} \in \mathbb{U}$. The sets $\langle \mathbb{U}, V \rangle = \{f \in C(X,Y) \mid f^{-1}(V) \in \mathbb{U}\}$ then form a subbase for the Isbell topology as $\mathbb{U}$ runs over all Scott-open subsets of $\mathcal{O}_X$ and $V \subseteq Y$ runs over all open subsets.

The Isbell topology contains the compact-open topology. For if $K \subseteq X$ is compact, then $\mathbb{U}_K = \{U \in \mathcal{O}_X \mid K \subseteq U\}$ is Scott-open and $\langle \mathbb{U}_K, V \rangle = \langle K, V \rangle$. To see that the Isbell topology is splitting take $f : Z \times X \to Y$ and consider its adjoint $\widetilde{f} : Z \to C_{Is}(X,Y)$. If $\widetilde{f}(z) \in \langle \mathbb{U}, V \rangle$, then there is $U \in \mathbb{U}$ such that $\widetilde{f}(z)(U) = f(z \times U) \subseteq V$ (use the fact that $\mathbb{U}$ is upwards closed). For each $x \in U$ there is a open set $U_x \subseteq U$ containing x and a neighbourhood $T_x \subseteq Z$ of z such that $f(T_x \times U_x) \subseteq V$. Since $\bigcup U_x = U \in \mathbb{U}$ there are a finite number of points $x_1, \ldots, x_n \in U$ such that $U_0 = U_{x_1} \cup \cdots \cup U_{x_n} \in \mathbb{U}$. Then $z \in T_0 = \bigcap_{i=1}^n T_{x_i}$ and $f(T_0 \times U_0) = \widetilde{f}(T_0)(U_0) \subseteq V$. That is $\widetilde{f}(T_0) \subseteq \langle \mathbb{U}, V \rangle$. Thus $\widetilde{f}$ is continuous.

(2) If (Y,d) is a metric space, then the topology of **uniform convergence** on $C(X,Y)$ is cosplitting. It is not in general splitting. The topology itself is that defined by the metric $\hat{d}(f,g) = \sup_{x \in X} \left(\frac{d(f(x),g(x))}{1+d(f(x),g(x))} \right)$. To illustrate take the multiplication $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(s,t) \mapsto s \cdot t$. The adjoint satisfies $\hat{d}(\widetilde{\mu}(s), \widetilde{\mu}(s')) = \sup_{t \in \mathbb{R}} \left(\frac{|s-s'||t|}{1+|s-s'||t|} \right) = 1$ for $s \neq s'$, and in particular is not continuous. The uniform topology always contains the compact-open toplogy. For compact domains the two topologies coincide, but even for the locally compact $\mathbb{R}$ they are different, since $C_{ko}(\mathbb{R},\mathbb{R})$ is exponential. Note that $C_{ko}(\mathbb{R},\mathbb{R})$ is even a metric topology ($\mathbb{R}$ is hemicompact).

(3) Define the **natural topology** on $C(X,Y)$ to be the initial topology induced by the family of all functions $Z \to C(X,Y)$ which are adjoint to continuous maps $Z \times X \to Y$. Let $C_{Nat}(X,Y)$ be the resulting space. The natural topology is evidently splitting. If σ is any topology on $C(X,Y)$, then the identity $C_{Nat}(X,Y) \to C_\sigma(X,Y)$ will be continuous if and only each composite $Z \to C_{Nat}(X,Y) \to C_\sigma(X,Y)$ is continuous for each map $Z \to C_{Nat}(X,Y)$ adjoint to a continuous map $Z \times X \to Y$. This will be the case if and only σ is splitting. Thus the natural topology on $C(X,Y)$ is the finest splitting topology. It also admits a second description: the natural topology on $C(X,Y)$ is the intersection of all cosplitting topologies. To see this let $P \subseteq C_{Nat}(X,Y)$ be any non-closed subset and let $(f_i)_{\mathcal{I}} \subseteq P$ be a net converging (necessarily continuously) to a function $f \notin P$. Let $\sigma(f)$ be the topology on $C(X,Y)$ such that $U \subseteq C(X,Y)$ is open if and only if either (i) $f \notin U$, or (ii) $f_i \in U$ for all sufficiently large i. The claim is that $\sigma(f)$ is a cosplitting topology in which $f_i \to f$. This will show that P is not open in $\bigcap$(cosplitting topologies) and hence that the natural topology is no stronger than this topology (it is already weaker). Details are left to the reader.

(4) Say that a net $(f_i)_{\mathcal{I}} \subseteq C(X,Y)$ **converges evenly** to f if for each open $V \subseteq Y$, each

point $y \in V$ has a neighbourhood $W_y \subseteq y$ such that $f^{-1}(W_y) \subseteq f_i^{-1}(V)$ for sufficiently large i. The **topology of even convergence** $C_{ev}(X,Y)$ is the topology generated by this convergence structure (i.e. $U \subseteq C_{ev}(X,Y)$ is open if and only if every net which converges evenly to a function in U is eventually in U). The topology of even convergence is our sole example of a cosplitting topology and will play no serious rôle in the sequel. □

In the third example above we see that in general there is no coarsest cosplitting topology. For instance $C(\mathbb{R}^\omega, I)$ carries no coarsest cosplitting topology. In any case we have a sequence of increasingly fine topologies on $C(X,Y)$.

$$\text{pointwise} \subseteq \text{compact-open} \subseteq \text{Isbell} \subseteq \text{natural} \subseteq \text{even convergence.} \tag{0.3}$$

If Y is Hausdorff, then each of these topologies is Hausdorff. In general each inclusion is strict. The first four are splitting, while the last is cosplitting. If any exponential topology exists, then it is the natural topology (in fact in this case the natural and Isbell topologies coincide, but this is not obvious). If X is locally compact, then the the compact-open topology is exponential, and hence coincides with the Isbell and natural topologies. Of course many topologies exist which are neither splitting nor cosplitting, and splitting and cosplitting topologies exist which are not comparable with the topologies in the above list.

Example 0.3 (1) Recall that the Baire space $\mathbb{N}^\omega$ is homemorphic to the irrationals $\mathbb{P}$ in their subspace topology inherited from $\mathbb{R}$. Thus $\mathbb{N}^\omega$ is not locally compact. It's known that the compact-open and Isbell topologies agree on $C(\mathbb{N}^\omega, \mathbb{N})$, and both are strictly weaker than the natural topology.

(2) Let $X = \mathbb{N} \cup \{\infty\}$. Topologise it by declaring a subset $U \subseteq X$ open iff $\infty \in U$. Then X is locally compact and consequently the compact-open, Isbell, and natural topologies on $C(X,Y)$ all coincide and are exponentiable for any space Y. On the other hand all compact subsets of X are finite, so the compact-open and pointwise topologies coincide. Still, the space is very badly behaved. It fails to be T_1, and in fact any map from X to a T_1 space is constant ($f(X) = f(\overline{\infty}) \subseteq \overline{f(\infty)}$). Thus whenever Y is T_1, the exponential Y^X is homeomorphic to Y.

Remark There is an alternative approach to the above results which uses convergence methods. Say that a net $(f_i)_{\mathcal{I}} \subseteq C(X,Y)$ **converges continuously** to $f \in C(X,Y)$ if for each convergent net $(x_j)_{\mathcal{J}} \to x$ in X, the net $(f_i(x_j))_{\mathcal{I}\times\mathcal{J}} \subseteq Y$ converges to $f(x)$. Now let σ be a topology on $C(X,Y)$. Then:

- σ is splitting if and only if continuous convergence implies σ-convergence.
- σ is cosplitting if and only if σ-convergence implies continuous convergence.
- σ is exponential if and only if σ-convergence is equivalent to continuous convergence.

The natural topology coincides with the topology generated by the continuously convergent nets. Thus an exponential topology exists when convergence in the natural topology is exactly the continuous convergence. □

Having set out the preliminaries, in the remainder of the talk we will answer the original question by establishing the following.

Theorem 0.6 *The following statements about a given space X are equivalent.*

(1) X is exponentiable.

(2) An exponential topology exists on $C(X, Y)$ for each space Y.

(3) X is core compact.

(4) An exponential topology exists on $C(X, \mathbb{S})$, where $\mathbb{S}$ is the Sierpinski space.

(5) If $q : Y \to Z$ is a quotient map, then $q \times id_X : Y \times X \to Z \times X$ is a quotient map.

The equivalence of the first two points has already been established. We will explain the third and fourth points below, and will carefully define the terms as we come to them. The last point we can explain now: the special adjoint functor theorem guarantees that the functor $(-) \times X$ will have a right adjoint if and only if it preserves all colimits (recall that Top is cocomplete, co-well-powered and has a seperating object. See Borceux Vol. 1 Th. 3.3.4). In turn $(-) \times X$ will preserve all colimits if and only if it preserves all coends, if and only if it preserves coproducts and coequalisers, if and only if it preserves coproducts and takes quotient maps to quotient maps.

A direct proof runs by testing the universal property of the quotient map. Assuming that X is exponentiable, a composite $Y \times X \xrightarrow{q \times id_X} Z \times X \xrightarrow{h} P$ will be continuous if and only if $Y \xrightarrow{q} Z \xrightarrow{\widetilde{h}} P^X$ is continuous if and only if $\widetilde{h}$ is continuous if and only if h is continuous. A third proof is also possible using the definition of *core compactness*.

Definition 4 *Let X be a space.*

*(1) Given subsets $A, B \subseteq X$, it is said that A is **well below** B[4], written $A \Subset B$, if $A \subseteq B$ and if from each open covering of B one can extract a finite subcovering of A.*

*(2) The space X is said to be **core compact** if whenever $x \in V \in \mathcal{O}_X$, there is $x \in U \in \mathcal{O}_X$ with $U \Subset V$.* □

Observe that A is compact if and only if $A \Subset A$. More generally, if $A \Subset B$ and A is closed, then A is compact. The closedness is necessary, for instance, as $(0, 1) \Subset [0, 1]$. In fact, if B is compact, then every subset $A \subseteq B$ is well below B. Note also that $A \Subset B \subseteq C \Subset D$ implies that $A \Subset D$.

Proposition 0.7 *Each locally compact space is core compact. A regular or Hausdorff space is core compact if and only if it is locally compact.*

[4]or **bounded** in B

Proof (1) If X is locally compact and $x \in V \in \mathcal{O}_X$, then there is compact $K \subseteq X$ with $x \in K^\circ \subseteq K \subseteq V$. Clearly $K^\circ \Subset V$. (2) Necessity is by the first part, so let X be core compact. We first assume that X is regular. Then whenever $x \in U \Subset V$, due to regularity, there is a closed neighbourhood C of x with $x \in C \subseteq U$. This gives $C \Subset V$, and hence implies that C is compact.

Now drop the regularity assumption and assume that X is core compact and Hausdorff. We will show that X is regular. For this it will suffice to show that each point has a local base of closed neighbourhoods, and the assmumption of core compactness reduces this to showing that $\overline{U} \subseteq V$ whenever $U \Subset V$ are given open sets. For this suppose $\overline{U} \not\subseteq V$. That is, that there is a point $y \in X \setminus V$, each of whose neighbourhoods meets U. Then because X is Hausdorff, for each $x \in V$ there are disjoint open sets P_x, Q_x with $x \in P_x \subseteq V$ and $y \in Q_x$. Because $U \Subset V$, there is thus a finite number of points $x_1, \ldots, x_n \in V$ such that $U \subseteq P = \bigcup_{i=1}^n P_{x_i}$. On the other hand $Q = \bigcap_{i=1}^n Q_{x_i}$ is a neighbourhood of y which is disjoint from P. Hence there is a contradiction, so it must be that $\overline{U} \subseteq V$. ▮

On the other hand not every core compact space is locally compact.

Example 0.4 *Let $Y = I \times [0, 1)$ be given the topology whose open sets are of the form $U_f = \{(x, y) \mid y < f(x)\}$, where $f : I \to I$ is a lower semicontinuous function. Fix a dense subset $A \subseteq I$ with the property that $A \cap U$ is not a Borel set for any nonempty open $U \subseteq I$. Now let $X = \{(x, y) \in Y \mid x \in A \Rightarrow y \in [0, 1) \cap \mathbb{Q},\ x \notin A \Rightarrow y \in [0, 1) \cap \mathbb{P}\}$. Then X is a second-countable core compact T_0 space. However every compact subset of X has an empty interior, so X is not even close to being locally compact.* □

We show next that every core compact space is exponentiable.

Lemma 0.8 *The following statements about a space X hold.*

(1) If $A_i \Subset B_i \subset X$ are subsets for $i = 1, 2$, then $A_1 \cup A_2 \Subset B_1 \cup B_2$.

(2) If X is core compact and $U, V \subseteq X$ are open sets with $U \Subset V$, then there is an open set $W \subseteq X$ with $U \Subset W \Subset V$.

Proof (1) Clear. (2) For each $x \in V$ iteratively choose open $\widetilde{W}_x, W_x \subseteq X$ such that $x \in \widetilde{W}_x \Subset W_x \Subset V$. The family $\{\widetilde{W}_x\}_{x \in V}$ covers V, so there are finitely many points $x_1, \ldots, x_n \in V$ such that $U \subseteq \bigcup_{i=1}^n W_{x_i}$. Using the first part of the lemma we can now write $U \subset \bigcup_{i=1}^n \widetilde{W}_{x_i} \Subset \bigcup_{i=1}^n W_{x_i} \Subset V$, so putting $W = \bigcup_{i=1}^n W_{x_i}$ we have $U \Subset W \Subset V$. ▮

Proposition 0.9 *If X is core compact, then for any space Y the Isbell topology on $C(X, Y)$ is exponential.*

Proof Since the Isbell topology is always splitting it will suffice to show the evaluation map $C_{Is}(X, Y) \times X \to Y$ is continuous. So let $(f, x) \in C_{Is}(X, Y) \times X$ and assume that $ev(f, x) = f(x) \in V \subseteq Y$ where V is open. Because X is core compact there is an open $U \subseteq X$ with $x \in U \Subset f^{-1}(V)$. Let $\mathbb{U} = \{W \in \mathcal{O}_X \mid U \Subset W\}$. The claim is that this set is Scott-open, and that $ev(\langle \mathbb{U}, V\rangle \times U) \subseteq V$. Because $(f, x) \in \langle \mathbb{U}, V\rangle \times U$ this implies that ev is continuous at (f, x).

The second part of the claim is immediate; since $\langle \mathbb{U}, V\rangle = \{g \in C_{Is}(X,Y) \mid U \Subset g^{-1}(V)\}$, if $g \in \langle \mathbb{U}, V\rangle$, then $g(U) \subseteq V$. As for the first part, $\mathbb{U}$ is nonempty since it contains $f^{-1}(V)$, and it is clearly upwards closed. Let $\{W_i\}_{\mathcal{I}} \subseteq \mathcal{O}_X$ be a family of open subsets with $\bigcup_{\mathcal{I}} W_i \in \mathbb{U}$. We have $U \Subset \bigcup_{\mathcal{I}} W_i$, so by Lemma 0.8 there is an open set P with $U \Subset P \Subset \bigcup_{\mathcal{I}} W_i$. It follows that there is a finite family $W_{i_1}, \ldots, W_{i_n}$ such that $U \Subset P \subset \bigcup_{k=1}^{n} W_{i_k}$. Hence $\bigcup_{k=1}^{n} W_{i_k} \in \mathbb{U}$, implying that $\mathbb{U}$ is Scott-open. ∎

Remark There is another topology on $C(X,Y)$ which is directly associated with the notion of core compactness. It is called the **core-open** topology and is generated by the subbase consisting of the sets $\langle U \Downarrow V\rangle = \{f \in C(X,Y) \mid U \Subset f^{-1}(V)\}$, where $U \subseteq X$ and $V \subseteq Y$ are open. The core-open topology contains the compact-open topology, but need not be comparable with the Isbell or natural topologies. In general it is neither splitting nor cosplitting. If X is core compact, however, then the core-open topology agrees with the Isbell topology (cf. the proof of Th. 0.9) and is thus the unique exponential topology on $C(X,Y)$. Of course in this case it coincides with the natural topology. Similarly, if X is locally compact, then the core-open and compact-open topologies are one and the same and agree with the other two mentioned topologies. □

We return now to our main theorem 0.6. We have shown the sufficiency of the third claim. Necessity will follow by understanding the fourth. For this we introduce the Sierpinski dyad, which is the space $\mathbb{S} = \{0,1\}$ with two points and topology $\mathcal{O}_{\mathbb{S}} = \{\emptyset, \mathbb{S}, \{1\}\}$. Then given any space X there is a bijection $C(X,\mathbb{S}) \xrightarrow{\cong} \mathcal{O}_X$, $f \mapsto f^{-1}(1)$. The inverse sends $U \in \mathcal{O}_X$ to its characteristic function

$$\chi_U : x \mapsto \begin{cases} 1 & x \in U \\ 0 & x \notin U. \end{cases} \tag{0.4}$$

Using this association, a high brow approach to the exponentiation problem reduces it to the existence of certain topologies on the lattice of open sets $\mathcal{O}_X$. One shows that a necessary condition for X to be exponential is that $\mathcal{O}_X$ is a *continuous lattice*. As it turns out this is exactly the condition that X be core compact. We will not pursue this path, preferring a cleaner topological approach using the material established above.

Fix a space X. For a subset $A \subseteq X$, as before we write $\langle A, \{1\}\rangle = \{f \in C(X,\mathbb{S}) \mid A \subseteq f^{-1}(1)\}$. In terms of the lattice of open sets this is the subset $\mathbb{U}_A = \{U \in \mathcal{O}_X \mid A \subseteq U\}$. Suppose given a family $\{W_i \subseteq X\}_{\mathcal{I}}$ of open subsets. Write $W = \bigcup_{\mathcal{I}} W_i$ and define a topology ω on $C(X,\mathbb{S})$ as follows. If $U \neq W$, then χ_U is isolated. The basic open neighbourhoods of χ_W are the sets of the form $\langle V_{i_1} \cup \cdots \cup V_{i_n}, \{1\}\rangle$, where $\{i_1, \ldots, i_n\} \subseteq \mathcal{I}$ is a finite subset and $V_{i_k} \subseteq W_{i_k}$ is an open subset for each $k = 1, \ldots, n$.

Lemma 0.10 *The topology ω on $C(X,\mathbb{S})$ is cosplitting.*

Proof Consider the evaluation map $ev : C_\omega(X,\mathbb{S}) \times X \to \mathbb{S}$. Its continuity is clear at all points except those of the form (χ_W, x). If $\chi_W(x) = 1$, then $x \in W = \bigcup_{\mathcal{I}} W_i$, and there is $i \in \mathcal{I}$ such that $x \in W_i$. Then $(\chi_W, x) \in \langle W_i, \{1\}\rangle \times W_i$ and $ev(\langle W_i, \{1\}\rangle \times W_i) = \{1\}$. ∎

Proposition 0.11 *Let X be a space. Assume that the set $C(X,\mathbb{S})$ carries an exponential topology σ. Then X is core compact.*

Proof Let $x \in U \in \mathcal{O}_X$. Then the continuity of the evaluation $ev : C_\sigma(X, \mathbb{S}) \times X \to \mathbb{S}$ lets us find an open neighbourhoods $\mathcal{N} \subseteq C_\sigma(X, \mathbb{S})$ of χ_U and $V \subseteq X$ of x such that $ev(\mathcal{N} \times V) = \{1\}$. Because $ev(\{\chi_U\} \times V) = \{1\}$ we have $V \subseteq U$. We claim that $V \Subset U$.

To verify this let $\mathcal{W} = \{W_i\}_{\mathcal{I}}$ be an open covering of U. Write $W = \bigcup_{\mathcal{I}} W_i$ and form the associated topology ω on $C(X, \mathbb{S})$ as discussed above. By the assumption that the given topology σ is exponential we have using Lemma 0.10 that $\sigma \subseteq \omega$. Thus there are indices $i_1, \ldots, i_k \in \mathcal{I}$ and open subsets $\widetilde{W}_{i_k} \subseteq W_{i_k}$, $k = 1, \ldots, n$, such that the ω-basic open set satisfies $\chi_U \in \langle \widetilde{W}_{i_1} \cup \cdots \cup \widetilde{W}_{i_k}, \{1\} \rangle \subseteq \mathcal{N}$. Immediately this gives $\bigcup_{k=1}^n \widetilde{W}_{i_k} \subseteq \bigcup_{k=1}^n \widetilde{W}_{i_k} \subseteq U$. On the other hand $ev(\langle \widetilde{W}_{i_1} \cup \cdots \cup \widetilde{W}_{i_k}, \{1\} \rangle \times V) = \{1\}$ implies that $V \subseteq \bigcup_{k=1}^n \widetilde{W}_{i_k}$. We may conclude that $V \Subset U$. ∎

With this established we have completed the proof of Theorem 0.6.

Remark Retain the assumptions of Propsition 0.11. Under the bijection $C(X, \mathbb{S}) \cong \mathcal{O}_X$, we can show that the open sets in the unique expontial topology are exactly the Scott-open subsets of the lattice $\mathcal{O}_X$. In particular the so-called Scott topology on $\mathcal{O}_X$ is the unique expontial topology, and this is exactly the Isbell topology on $C(X, \mathbb{S})$. Of course this is also the natural topology. □

Outlook:

Definition 5 *Let Y be a fixed space.*

*(1) A space X is said to be Y-**consonant** if the compact-open and Isbell topologies on $C(X, Y)$ coincide.*

*(2) A space X is said to be Y-**concordant** if the natural and Isbell topologies on $C(X, Y)$ coincide.*

*(3) A space X is said to be Y-**harmonic** if it is both Y-consonant and Y-concordant.*

*The space X is said to be **consonant** if it is Y-consonant for each space Y, and **concordant** if it is Y-concordant for each space Y.* □

Proposition 0.12 *The following statements about a space X are equivalent.*

(1) X is consonant.

(2) X is both $\mathbb{S}$-consonant and $\mathbb{S}$-concordant.

(3) X is $\mathbb{S}$-consonant.

If X is Tychonoff then the conditions $(1) - (3)$ above are equivalent to X being $\mathbb{R}$-consonant.
∎

Eg. The Sorgenfrey line is not consonant. $\mathbb{Q}$ is not consonant. Current research is interested in understanding and characterising the consonant and concordant spaces. For instance it is know that every Čech complete Tychonoff space is consonant. For example the irrationals $\mathbb{P} \cong \mathbb{N}^\omega$ are consonant but not concordant.